POWER FUNCTION OF THE LIKELIHOOD RATIO TEST WHEN RANGE DEPENDS UPON THE PARAMETER

BY M. C. JAISWAL AND C. G. KHATRI
Gujarat University
(Received in October, 1967)

SUMMARY

Hogg, R.V. [1] obtained the null distribution of likelihood ratio test statistics for testing the hypothesis $H_0 : (\theta_1=\theta_2=\ldots\theta_k=\theta_0$ given) and $H_o : (\theta_1=\theta_2=\ldots\theta_k)$ for certain non-regular densities given by

$$f(x, \theta_i)=g(x)/h(\theta_i) \text{ for } a \leqslant x \leqslant \theta_i$$

for $i=1, 2, \ldots k$ and $h(\theta_i)=\int_a^{\theta_i} g(x)\, dx$

The non-null distributions of the likelihood ratio test statistics for testing $H_0 : (\theta_1=\theta_2\ldots\theta_k=\theta_0$ given) for any k and $H_0 : (\theta_1=\theta_2=\ldots\theta_k)$ for $k=2, 3$ and 4 have been obtained and conjectured for any k. These results hold true if the range of x is $\{\theta_i, b\,\theta_i\}$, where $b(\theta)$ is a strictly monotone decreasing continuous function of θ.

1. INTRODUCTION

The asymptotic distribution of the likelihood ratio test statistic depends essentially on the regularity conditions, as shown by Wilks [3], which are necessary to establish the asymptotic normality of maximum likelihood estimator. These conditions are not satisfied when the range depends upon the parameters. As for example, the density function given by

$$(1) \qquad f(x, \theta)=g(x)/h(\theta) \text{ for } a \leqslant x \leqslant \theta$$
$$=0 \qquad \text{otherwise,}$$

where $h(\theta)=\int_a^{\theta} g(x)\, dx$ is a monotone continuous function of θ, does not satisfy the regularity conditions. Hogg R.V. [1] and Kendall and

Stuart [2] have derived the likelihood ratio test statistics l_1 and l_2 for testing

(i) $H_0 : (\theta_1=\theta_2=...=\theta_k=\theta_0$ given) and

(ii) $H_0 : (\theta_1=\theta_2=...=\theta_k)$

respectively, considering the $k(k=1, 2,...)$ mutually independent populations having density functions $f(x, \theta_i)$, $i=1, 2,...k$. It has been shown by them that under the respective null hypothesis $(-2 \log l_t)$ is distributed as x^2 with $2(k-t+1)$ degrees of freedom, for $t=1, 2$.

Let x_{ij} $(j=1, 2,...n_i, i=1, 2...k)$ be independent observations from the population given by the density function $f(x, \theta_i)$, $i=1, 2...k$, and $x_{(i)}$ be the largest value of the ith sample. The likelihood ratio test statistics l_1 and l_2 can be written as

$$(2) \qquad l_1=\prod_{i=1}^{k}\{h(x_{(i)})/h(\theta_0)\}^{n_i}$$

and

$$(3) \qquad l_2=\prod_{i=1}^{k}\{h(x_{(i)})/h(m)\}^{n_i},$$

where $\qquad m=\max.(x_{(1)}, x_{(2)},...x_{(k)})$

Moreover, we note that the density function of $y_i=h(x_{(i)})$ (see Hogg [1]) can be shown to be

$$(4) \qquad n_i\, \phi_i^{-n_i}\, y_i^{n_i-1} \quad \text{for } o \leqslant y_i \leqslant h(\theta_i)=\phi_i$$

and

$$o \qquad \text{otherwise}$$

The purpose of this paper is to obtain the exact non-null distribution of l_1 and l_2. It is shown that the non-null distribution of $-2 \log \frac{l_1}{G}$, where G is given by

$$(5) \qquad G=\prod_{i=1}^{k}\{h(\theta_i)/h(\theta_0)\}^{n_i}$$

is distributed as x^2 with $2k$ degrees of freedom. This result has been proved by Kendall and Stuart [2] for $k=1$ explicitly, and its extension for any k is immediate. The non-null distribution of l_2 is obtained for $k=2$, 3 and 4 only and conjectured for any k. These results hold true even if the range of x is $(\theta_i, b(\theta_i)$ where $b(\theta)$ is strictly a monotone decreasing continuous function of θ. This has been established for the null case by Kendall and Stuart [2].

2. Non-null Distribution of l_1

The likelihood ratio test statistic for testing $H_0(\theta_1=\theta_2=\ldots\theta_k=\theta_0$ given) given by (2) can be rewritten as

(6) $$l_1=Gl_{1,0}$$

where G is given by (5) and

$$l_{1,0}=\prod_{i=1}^{k} w_i^{n_i} \text{ with } w_i=y_i/h(\theta_i),$$

$i=1, 2,\ldots k$, which are independently distributed with the density functions

(7) $$n_i \; w_i^{n_i-1} \quad \text{for } 0 \leqslant w_i \leqslant 1$$

and $\quad$ 0 otherwise

It is easy to see from Hogg [1] and Kendall and Stuart [2 p. 237] that $-2\log l_{i,0}$ is distributed as x^2 with $2k$ degrees of freedom. Hence, $-2\log\frac{l_1}{G}$ is distributed as x^2 with $2k$ degrees of freedom.

3. Power Function of l_2

3·1. Power Function when k=2

Let $k=2$ in (3), then l_2 will be given by

(8) $$y_1^{n_1}\, y_2^{n_2} / y^{n_1+n_2}$$

where $\quad y=\max.(y_1, y_2)$ and $y_i=h(x_{(i)})$

Let $\quad \alpha_{12}=\phi_1/\phi_2$ and

$$L(n_1, n_2, c\,;\alpha_{12})=P(l_2 \leqslant c)$$

(9) $$=P[(y_1/y_2)^{n_1} \leqslant c \text{ when } y_2 \geqslant y_1]+$$

$$P[y_2/y_1)^{n_2} \leqslant c \text{ when } y_1 \geqslant y_2]$$

because the event $l_2 \leqslant c$ is the union of two mutually exclusive cases

$$y_1^{n_1} \leqslant c\, y_2^{n_1}, \text{ when } y_2 \geqslant y_1$$

and

$$y_2^{n_2} \leqslant c\, y_1^{n_1} \text{ when } y_1 \geqslant y_2$$

From (9) it is obvious that

(10) $L(n_1, n_2, c\,;\, \alpha_{12}) = L(n_2;\, n_1, c\,;\, \alpha_{21})$

Hence, we shall obtain $L\,(n_1, n_2, c\,;\, \alpha_{12})$ when

$$\alpha_{12} \leqslant 1 \quad i.e. \quad \phi_1 \leqslant \phi_2$$

Let us denote the density function of l_2 at $l_2 = c$ by $f(c)$ then

(11) $f(c) = f_1(c) = f_2(c)$

where

(12) $f_1(c) = \dfrac{d}{dc}\, P\left[\; y_2^{n_1} \leqslant c\, y_2^{n_1}, \text{ when } y_2 \geqslant y_1 \right]$

and

(13) $f_2(c) = \dfrac{d}{dc}\, P\left[\; y_2^{n_2} \leqslant c\, y_1^{n_2}, \text{ when } y_1 \geqslant y_2 \right]$

The joint density function of y_1 and y_2 is

(14) $\dfrac{n_1\, n_2}{\phi_1^{n_1}\, \phi_2^{n_2}}\, y_1^{n_1-1}\, y_2^{n_2-1}$ for $0 < y_i \leqslant \phi_i,\ i = 1, 2.$

and 0 otherwise

From (14), we get the density function of $v_1 = \dfrac{y_1}{y_2}$ as

(15) $\dfrac{n_1\, n_2}{n_1 + n_2}\, \alpha_{21}^{n_1}\, v_1^{n_1-1}$ for $0 < v_1 \leqslant \alpha_{12}$,

$\dfrac{n_1\, n_2}{n_1 + n_2}\, \alpha_{12}^{n_2}\, v_1^{-n_2-1}$ for $\alpha_{12} < v_1 \leqslant \infty$

and 0 otherwise

From (15) it is easy to see that

(16) $f_1(c) = \dfrac{n_2}{n_1 + n_2}\, \alpha_{21}^{n_1}$ for $0 < c \leqslant \alpha_{12}^{n_1}$

$= \dfrac{n_2}{n_1 + n_2}\, \alpha_{12}^{n_2}\, c^{-1-\frac{n_2}{n_1}}$ for $\alpha_{12}^{n_1} > c \leqslant 1$

and

(17) $f_2(c) = \dfrac{n_1}{n_1 + n_2}\, \alpha_{12}^{n_2}$ for $0 < c \leqslant 1$

using (16) and (17) in (11) we get the density function of l_2 at $l_2 = c$ as

$$(18)\quad f(c)=(n_1\alpha_{12}^{\,n_2}+n_2\alpha_{21}^{\,n_1})\,/\,(n_1+n_2) \text{ for } 0<c\leqslant \alpha_{12}^{\,n_1}$$

$$=n_2\,\alpha_{12}^{\,n_2}\left(1+\;c^{-1-\frac{n_2}{n_1}}\right)\,/\,(n_1+n_2), \text{ for } \alpha_{12}^{\,n_1}<c\leqslant 1$$

$$=0 \qquad \text{otherwise}$$

Hence

$$(19)\quad L(n_1,\, n_2,\, c\,;\alpha_{12})=c\left(n_1\,\alpha_{12}^{\,n_2}+n_2\alpha_{21}^{\,n_1}\right)/(n_1+n_2)$$

$$\text{for } o<c\leqslant\alpha_{12}^{\,n_1}$$

$$=1-[n_1/(n_1+n_2)]\alpha_{12}^{\,n_2}\,c^{-\frac{n_2}{n_1}}$$

$$\text{for } \alpha_{12}^{\,n_1}<c\leqslant 1$$

$$=1 \qquad \text{for} \quad c>1$$

$$=0 \qquad \text{for} \quad c<0.$$

3.2 Power Function when K=3.

Let $k=3$ in (3), then l_2 will be given by

$$(20)\qquad y_1^{\,n_1}\,y_2^{\,n_2}\,y_3^{\,n_3}\,/y^{\,n_1+n_2+n_3}$$

where $y=\max(y_1,\, y_2,\, y_3)$

and $y_i=h(x_{(i)})$

Let $\alpha_{ij}=\phi_i/\phi_j$, $N_i=\sum_{j=1}^{i} n_j$, $N_0=0$,

$$B_2=\alpha_{12}^{\,n_1},\quad B_3=\alpha_{13}^{\,n_1}\alpha_{23}^{\,n_2}.$$

$$(21)\qquad A_{i,\,3}=\prod_{j=1}^{3}\alpha_{i_j}^{\,n_j}\quad i=1,\,2,\,3$$

and let

$$L(n_1,\, n_2,\, n_3,\, c\,;\,\alpha_{12},\,\alpha_{13},\,\alpha_{23})$$

$$=P(l_2\leqslant c)$$

$$(22)\quad =P[y_2^{\,n_2}\,y_3^{\,n_3}\leqslant c\;\;y_1^{\,n_2+n_3} \qquad \text{when } y_1\geqslant(y_2,\, y_3)]$$

$$+P[y_1^{\,n_1}\,y_3^{\,n_3}\leqslant c\;\;y_2^{\,n_1+n_3} \qquad \text{when } y_2\geqslant(y_1,\, y_3)]$$

$$+P[y_1^{\,n_1}\,y_2^{\,n_2}\leqslant c\;\;y_3^{\,n_1+n_2} \qquad \text{when } y_3\geqslant(y_1,\, y_2)]$$

because the event $l_2 \leqslant c$ is the union of the three mutually exclusive events given by

$$A_1 : y_2^{n_2} y_3^{n_3} \leqslant c \; y_1^{n_2+n_3} \qquad \text{when } y_1 \geqslant (y_2, y_3)$$

$$A_2 : y_1^{n_1} y_3^{n_3} \leqslant c \; y_3^{n_1+n_3} \qquad \text{when } y_2 \geqslant (y_1, y_3)$$

(23) and

$$A_3 : y_1^{n_1} y_2^{n_2} \leqslant c \; y_2^{n_1+n_2} \qquad \text{when } y_3 \geqslant (y_1, y_2)]$$

Let the density function of l_2 at $l_2=c$ be denoted by $f(c)$ then

(24) $$f(c)=f_1(c)+f_2(c)+f_3(c)$$

where $$f_i(c)=\frac{d}{dc} P(A_i),$$

$$i=1, 2, 3.$$

Moreover, it is easy to see from (22) that

(25) $$L(n_1, n_2, n_3, c\,;\alpha_{12}, \alpha_{13}, \alpha_{23}) = L(n_{i_1}, n_{i_2}, n_{i_3}, c\,;\alpha_{i_1 i_2}, \alpha_{i_1 i_3}, \alpha_{i_2 i_3})$$

where (i_1, i_2, i_3) is any permutation of (1, 2, 3).

Hence, we shall obtain $L(n_1, n_2, n_3, c\,;\alpha_{12}, \alpha_{13}, \alpha_{23})$ when $\phi_1 \leqslant \phi_2 \leqslant \phi_3$ *i.e.* $\alpha_{12} \leqslant 1$, $\alpha_{13} \leqslant 1$ and $\alpha_{23} \leqslant 1$.

Now the joint density function of y_1, y_2 and y_3 is given by

(26) $$\frac{n_1 n_2 n_3}{\phi_1^{n_1} \phi_2^{n_2} \phi_3^{n_3}} \, y_1^{n_1-1} \, y_2^{n_2-1} \, y_3^{n_3-1}$$

and $\qquad$ for $0 < y_i \leqslant \phi_i$, $i=1, 2, 3$

0 $\qquad$ otherwise

From (26), after deriving the joint density function of

$$v_2=y_2/y_1$$

and $$v_3=y_3/y_1$$

we get $f(c)$ from it as

(27) $$f_1(c)=(n_1/N_3)\alpha_{12}^{n_2}\alpha_{13}^{n_3} (-\log c)$$

[for $0 < c \leqslant 1$.

Similarly after deriving the joint density function of

$$v_4=\frac{1}{v_2}$$

and $$\nu_5=\frac{\nu_3}{\nu_2}$$

we get $f_2(c)$ from it as

$$(28)\qquad f_2(c)=\frac{n_2}{N_2}\left(-\log\frac{c}{B_2}\right)A_{2,3}+\frac{n_1n_2}{N_3{}^2}(A_{2,3}-A_{1,3}) \quad \text{for } 0<c\leqslant B_2$$

$$=\frac{n_1n_2}{N_3{}^2}\left(c^{-\frac{N_3}{N_1}}-1\right)A_{1,3} \quad \text{for } B_2<c\leqslant 1$$

$$=0 \qquad \text{otherwise}$$

Lastly, after deriving the joint density function of

$$\nu_6=\frac{1}{\nu_3}$$

and $$\nu_7=\frac{\nu_2}{\nu_3}$$

we get $f_3(c)$ from it as

(29) $f_3(c)$

$$=\frac{N_3-N_2}{N_3}\left(-\log\frac{c}{B_3}\right)A_{3,3}+\frac{(N_1+N_2)(N_3-N_2)}{N_3{}^2}(A_{3,3}-A_{2,3})$$
$$+\frac{N_1(N_3-N_2)}{N_3{}^2}(A_{2,3}-A_{1,3}) \quad \text{for } 0<c\leqslant B_3$$

$$=\frac{(N_1+N_2)(N_3-N_2)}{N_3{}^2}\left\{\left(\frac{c}{B_2}\right)^{-\frac{N_3}{N_2}}-1\right\}A_{2,3}$$
$$+\frac{N_1(N_3-N_2)}{N_3{}^2}(A_{2,3}-A_{1,3}) \quad \text{for } B_3\leqslant c\leqslant B_2$$

$$=\frac{N_1(N_3-N_2)}{N_3{}^2}\left(c^{-\frac{N_3}{N_1}}-1\right)A_{1,3} \quad \text{for } B_2\leqslant c\leqslant 1$$

Using (27), (28) and (29) in (24) we get the density function of l_2 at $l_2=c$ as

(30) $f(c)$

$$=\sum_{i=1}^{3}\frac{N_i-N_{i-1}}{N_3}\left(-\log\frac{c}{B_i}\right)A_{i,3}$$
$$+\sum_{i=1}^{3}\{(N_i/N_3)^2-(N_{i-1}/N_3)^2-(N_i/N_3)+(N_{i-1}/N_3)\}A_{i,3}$$
$$\text{for } 0\leqslant c\leqslant B_3$$

$$=\sum_{i=1}^{2} \frac{N_i-N_{i-1}}{N_3}\left(-\log \frac{c}{B_i}\right)A_{i,3}$$

$$+\sum_{i=1}^{2} \{(N_i/N_3)^2-(N_{i-1}/N_3)^2-(N_i/N_3)+(N_{i-1}/N_3)\}A_{i,3}$$

$$+\frac{N_2(N_3-N_2)}{N_3^2}\left(\frac{c}{B_2}\right)^{-\frac{N_3}{N_2}} A_{2,3}$$

for $B_3 < c \leqslant B_2$

$$=(N_1/N_3)(-\log c)A_{1,3}+\{(N_1/N_3)^2-(N_1/N_3)\}A_{1,3}$$

$$+\frac{N_1(N_3-N_1)}{N_3^2}c^{-\frac{N_3}{N_1}}A_{1,3}$$

for $B_2 < c \leqslant 1$

and

$=0$ otherwise

From (35) we get

$$L(n_1, n_2, n_3, c\ ; \alpha_{12}, \alpha_{13}, \alpha_{23})=$$

(31)

$$(-\log c)\left\{\sum_{i=1}^{3}(N_i-N_{i-1})A_{i3}\right\}+\frac{c}{N_3}\left\{\sum_{i=1}^{3}(N_i-N_{i-1})(\log B_i)A_{i,3}\right.$$

$$\left.+\frac{1}{N_3}\sum_{i=1}^{3}(N_i^2-N_{i-1}^2)A_{i,3}\right\}$$

for $0 < c \leqslant B_3$

$$\frac{(-c\log c)}{N_3}\left\{\sum_{i=1}^{2}(N_i-N_{i-1})A_{i,3}\right\}+\frac{c}{N_3}\left\{\sum_{i=1}^{2}(N_i-N_{i-1})A_{i,3}\right.$$

$$(\log B_i)$$

$$\left.+\frac{1}{N_3}\sum_{i=1}^{2}(N_i^2-N_{i-1}^2)A_{i,3}\right\}+1-\frac{N_2^2}{N_3^2}A_{2,3}\,B_2^{\frac{N_3}{N_2}}c^{1-\frac{N_3}{N_2}}$$

for $B_3 \leqslant c \leqslant B_2$

$$(-c\log c)\ \frac{N_1}{N_3}A_{1,3}+\frac{N_1^2}{N_3^2}A_{1,3}+1-\frac{N_1^2}{N_3^2}A_{1,3}\,c^{1-\frac{N_3}{N_1}}$$

for $B_2 \leqslant c \leqslant 1$

3.3 Power Function when k=4.

Let $\alpha_{ij}=\phi_i/\phi_j,\ A_{i,4}=\prod_{j=1}^{4}\alpha_{ij}^{nj}$

$$B_i=\prod_{j=1}^{i} \alpha_{ij}^{n_j}, \quad i, j=1, 2, 3, 4$$

$$B_5=0, \quad N_i=\sum_{j=1}^{i} n_j,$$

$$N_0=0, \quad i, j=1, 2, 3, 4$$

and

(32)

$$\begin{aligned} L(n_1, n_2, n_3, n_4, c\,;\, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) &= P(l_2 \leqslant c) \\ &= P[y_2^{n_2}\, y_3^{n_3}\, y_4^{n_4} \leqslant c\, y_1^{n_2+n_3+n_4}, \quad \text{when } y_1 \geqslant (y_2, y_3, y_4)] \\ &+ P[y_1^{n_1}\, y_3^{n_3}\, y_4^{n_4} \leqslant c\, y_2^{n_1+n_3+n_4}, \quad \text{when } y_2 \geqslant (y_1, y_3, y_4)] \\ &+ P[y_1^{n_1}\, y_2^{n_2}\, y_4^{n_4} \leqslant c\, y_3^{n_1+n_2+n_4}; \quad \text{when } y_3 \geqslant (y_1, y_2, y_4)] \\ &+ P[y_1^{n_1}\, y_2^{n_2}\, y_3^{n_3} \leqslant c\, y_4^{n_1+n_2+n_3}, \quad \text{when } y_4 \geqslant (y_1, y_2, y_3)] \end{aligned}$$

because the event $l_2 \leqslant c$ is the union of four mutually exclusive events given by

$$A_1 : y_2^{n_2}\, y_3^{n_3}\, y_4^{n_4} \leqslant c\, y_1^{n_2+n_3+n_4}, \quad \text{when } y_1 \geqslant (y_2, y_3, y_4)]$$

$$A_2 : y_1^{n_1}\, y_3^{n_3}\, y_4^{n_4} \leqslant c\, y_2^{n_1+n_3+n_4}, \quad \text{when } y_2 \geqslant (y_1, y_3, y_4)]$$

$$A_3 : y_1^{n_1}\, y_2^{n_2}\, y_4^{n_4} \leqslant c\, y_3^{n_1+n_2+n_4}, \quad \text{when } y_3 \geqslant (y_1, y_2, y_4)]$$

(33) and

$$A_4 : y_1^{n_1}\, y_2^{n_2}\, y_3^{n_3} \leqslant c\, y_4^{n_1+n_2+n_3}, \quad \text{when } y_4 \geqslant (y_1, y_2, y_3)]$$

Let us denote the density function $f(c)$ of l_2 at $l_2=c$ as

(34) $$f(c)=f_1(c)+f_2(c)+f_3(c)+f_4(c)$$

where

$$f_i(c)=\frac{d}{dc} P(A_i),$$

$$i=1, 2, 3, 4$$

Moreover, it is easy to see from (32) that

(35) $$\begin{aligned} L(n_1, n_2, n_3, n_4, c\,;\, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}) &= L(n_{i_1}, n_{i_2}, n_{i_3}, n_{i_4}, c\,;\, \ldots\ldots\ldots\ldots\ldots\ldots \\ &\alpha_{i_1 i_2}, \alpha_{i_1 i_3}, \alpha_{i_1 i_4}, \alpha_{i_2 i_3}, \alpha_{i_2 i_4}, \alpha_{i_3 i_4}) \end{aligned}$$

where i_1, i_2, i_3, i_4 is any permutation of (1, 2, 3, 4)

Hence, we shall obtain

$$L(n_1, n_2, n_3, n_4, c\ ;\ \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$$

when $$\phi_1 \leqslant \phi_2 \leqslant \phi_3 \leqslant \phi_4,$$

i.e., $$\underset{i<j}{\alpha_{ij}} \leqslant 1,\ i, j=1, 2, 3, 4$$

Proceeding in the same way as in section (3·1) and (3·2) we get

(36)

$$f_1(c)=\frac{n_1}{2N_4}\ (-\log c)^2\ A_{1,\,4}$$

for $0 < c \leqslant 1$

(37)

$$f_2(c)=\frac{n_1n_2}{N_4^2}\left(\log\ c-\ \frac{N_1}{N_4}\right)A_{1,\,4}$$

$$+\frac{n_2}{N_4}\left\{\frac{1}{2}\left(-\log\frac{c}{B_2}\right)^2-\frac{N_1}{N_4}\ \log\left(\frac{c}{B_2}\right)+\frac{N_1^2}{N_4^2}\right\}A_{2,4}$$

for $0 < c \leqslant B_2$

$$=\frac{n_1n_2}{N_4^2}\left(\log c-\frac{N_1}{N_4}\right)A_{1,\,4}+\frac{n_2N_1^2}{N_4^2}\,c^{-\frac{N_4}{N_1}}$$

for $B_2 < c < 1$

(38)

$$f_3(c)=\frac{n_1n_3}{N_4^2}\left\{\log c-\frac{N_1}{N_4}\right\}A_{1,\,4}+\frac{n_2n_3}{N_4^2}\left\{\log\frac{c}{B_2}-\frac{N_1}{N_4}-\frac{N_2}{N_4}\right\}A_{2,\,4}$$

$$+\frac{n_3}{N_4}\left\{\frac{1}{2}\left(\log\frac{c}{B_3}\right)^2-\frac{N_2}{N_4}\ \log\frac{c}{B_3}+\frac{N_2^2}{N_4^2}\right\}A_{3,\,4}$$

for $0 < c \leqslant B_3$

$$=\frac{n_1n_3}{N_4^2}\left(\log c-\frac{N_1}{N_4}\right)A_{1,\,4}+\frac{n_2n_3}{N_4^2}\left\{\log\frac{c}{B_2}-\frac{N_1}{N_4}-\frac{N_2}{N_4}\right\}A_{2,\,4}$$

$$+\frac{n_3N_2^2}{N_4^2}\ (c/B_2)^{-\frac{N_4}{N_2}}A_{2,\,4}$$

for $B_3 < c \leqslant B_2$

$$=\frac{n_1n_3}{N_4^2}\left(\log c-\frac{N_1}{N_4}\right)A_{1,4}+\frac{n_3N_1^2}{N_4^2}\ c^{-\frac{N_4}{N_1}}A_{1,\,4}$$

for $B_2 < C \leqslant 1$

and

$$(39)\quad f_4(c)=\frac{n_1n_4}{N_4^2}\left\{\log c-\frac{N_1}{N_4}\right\}A_{1,4}+\frac{n_2n_4}{N_4^2}\left\{\log\frac{c}{B_2}-\frac{N_1}{N_4}-\frac{N_2}{N_4}\right\}A_{2,4}$$

$$+\frac{n_3n_4}{N_4^2}\left\{\log\frac{c}{B_3}-\frac{N_2}{N_4}-\frac{N_3}{N_4}\right\}A_{3,4}$$

$$+\frac{n_4}{N_4}\left\{\tfrac{1}{2}\left(\log\frac{c}{B_4}\right)^2-\frac{N_3}{N_4}\log\frac{c}{B_3}+\frac{N_3^2}{N_4^2}\right\}A_{4,4}$$

for $0<c\leqslant B_4$

$$=\frac{n_1n_4}{N_4^2}\left\{\log c-\frac{N_1}{N_4}\right\}A_{1,4}+\frac{n_2n_4}{N_4^2}\left\{\log\frac{c}{B_2}-\frac{N_1}{N_4}-\frac{N_2}{N_4}\right\}A_{2,4}$$

$$+\frac{n_3n_4}{N_4^2}\left\{\log\frac{c}{B_3}-\frac{N_2}{N_4}-\frac{N_3}{N_4}\right\}A_{3,4}$$

$$+\frac{n_4N_3^2}{N_4^2}\left(\frac{c}{B_3}\right)^{-\frac{N_4}{N_3}}A_{3,4}$$

for $B_4<c\leqslant B_3$

$$=\frac{n_1n_4}{N_4^2}\left\{\log c-\frac{N_1}{N_4}\right\}A_{1,4}+\frac{n_2n_4}{N_4^2}\left\{\log\frac{c}{B_2}-\frac{N_1}{N_4}-\frac{N_2}{N_4}\right\}A_{2,4}$$

$$+\frac{n_4N_2^2}{N_4^2}\left(\frac{c}{B_2}\right)^{-\frac{N_4}{N_2}}A_{2,4}$$

for $B_3<c\leqslant B_2$

$$=\frac{n_1n_4}{N_4^2}\left\{\log c-\frac{N_1}{N_4}\right\}A_{1,4}$$

$$+\frac{n_4N_1^2}{N_4^2}\,c^{-\frac{N_4}{N_1}}A_{1,4}$$

for $B_2<c\leqslant 1$

Using (36), (37), (38) and (39) in (34) we get the density function of l_2 at $l_2=c$

$$(40)\quad f(c)=\left[\sum_{r=0}^{2}\sum_{i=1}^{j}\left\{\left(\frac{N_i}{N_4}\right)^{r+1}-\left(\frac{N_{i-1}}{N_4}\right)^{r+1}-\left(\frac{N_i}{N_4}\right)^{r}+\left(\frac{N_{i-1}}{N_4}\right)^{r}\right\}A_{i,k}\left(-\log\frac{c}{B_i}\right)^2\Big/(2-r)!\right]$$

$$+\frac{(N_4-N_j)N_j^2}{N_4^2}A_{j,4}\left(\frac{c}{B_j}\right)^{-\frac{N_4}{N_j}}$$

for $B_{j+1}<c\leqslant B_j \quad j=1,2,3,4$

from (40) we get

(41) $L(n_1, n_2, n_3, n_4, c\,;\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$

$$=\left[c\sum_{p=0}^{2}\sum_{i=1}^{j}\left\{\left(\frac{N_i}{N_4}\right)^{p+1}-\left(\frac{N_{i-1}}{N_4}\right)^{p+1}\right\}A_{i,4}\left(-\log\frac{c}{B_i}\right)^{2-p}\Big/(2-p)\,!\right]$$

$$+1-(N_j/N_4)^3\,A_{j\cdot 4}\,B_j^{\frac{N_4}{N_j}}\,c^{1-\frac{N_4}{N_j}}$$

for $B_{j+1}<c\leqslant B_j,\ j=1,\ 2,\ 3,\ 4$

From this we can conjecture the density function of l_2 at $l_2=c$ for any k as

(42) $$\left[\sum_{r=0}^{k-2}\sum_{i=1}^{j}\{N_i/N_k)^{r+1}-(N_{i-1}/N_k)^{r+1}-(N_i/N_k)^r+(N_{i-1}/N_k)^r\}A_{i,k}\right.$$

$$\left.(-\log\,(c/B_i))^{k-2-r}/(k-2-r)\,!\ \right]$$

$$+\{(N_k-N_j)N_j^{k-2}/N_k^{k-1}\}\,A_{j,k}\,(B_j/l_2)^{\frac{N_4}{N_j}},$$

for $B_{j+1}<c\leqslant B_j\quad j=1,2,\ldots k.$

where $B_{k+1}=0,\qquad \alpha_{ij}=\phi_i/\phi_j,\qquad i,j=1,\ 2,\ldots k$

$$B_i=\prod_{j=1}^{i}\alpha_{ji},\qquad A_{ik}=\prod_{j=1}^{k}\alpha_{ij}^{nj},\qquad i=1,\ 2,\ldots k$$

$$N_i=\sum_{j=1}^{i}n_j,\ N_0=0 \text{ and } 0<\phi_1\leqslant\phi_2\leqslant\phi_3\leqslant\ldots\ldots\leqslant\phi_k.$$

Hence, the power function of l_2 for any k is conjectured as

(43) $$\left[c\sum_{p=0}^{k-2}\sum_{i=1}^{j}\{(N_i/N_k)^{p+1}-(N_{i-1}/N_k)^{p+1}\}A_{i,k}\right.$$

$$\left.(-\log\,(c/B_i)\,)^{k-2-p}/(k-2-p)\,!\right]$$

$$+1-(N_j/N_k)^{k-1}\;A_{j,k}\;B_j^{N_k/N_j}\;c^{1-(N_k/N_j)}$$

for $B_{j+1}<c\leqslant B_J\,,\ j=1,\ 2,\ \ldots k.$

We have $P_r(B_{k+1}=0)=0$ for C $(\log c)^j\to 0$ as $C\to 0$ for any non-negative integer number j and $P_r\ (B_1=1)=1$

4. BOTH EXTREMITIES OF THE RANGE DEPENDING UPON θ.

We consider the density function

(44) $f(x, \theta_i)=g(x)/k(\theta_i)$ for $\theta_i \leqslant x \leqslant b(\theta_i)$

where $k(\theta_i)=\int_{\theta_i}^{b(\theta_i)} g(x)dx$, and $b(\theta_i)$ ($i=1, 2,...k$)

are strictly monotone decreasing function of θ. Hogg [1] has given the likelihood ratio test for testing $H_0 : (\theta_1=\theta_2=\ldots\theta_k=\theta_0$ given)

$$(45) \qquad l_3=\prod_{i=1}^{k} \{k(t_i)/k(\theta_0)\}^{n_i}$$

where x_{ij}, $j=1,2,\ldots, n_i$ are independent observations from $f(x, \theta_i)$, $i=1,2,\ldots k$.

let $t_i=min[\underset{j}{min}(x_{ij}), b(\underset{j}{max}\ x_{ij})]$, $i=1,2,\ldots k$

According to Hogg [1]; we get

$$P(t \geqslant r)=P\{\underset{j}{min}\ x_{ij} \geqslant r, \underset{j}{max}\ x_{ij} \leqslant b(r)\}$$

$$=\left(\frac{1}{k(\theta_i)}\right)^{n_i}\left(\int_r^{b(r)} g(x)dx\right)^{n_i}=\left[\frac{k(r)}{k(\theta_i)}\right]^{n_i}$$

Hence, it is essay to see that the density function of

$$y_i=k(t_i)$$

is given by

$$(46) \qquad n_i\phi_i^{-n_i}y_i^{n_i-1} \text{ for } 0 \leqslant y_i \leqslant \phi_i=k(\theta_i)$$

and 0 otherwise

This is the same density function as considered in the preceding sections and hence l_3 can be written as

$$(47) \qquad l_3=G_1\ l_{1,0}$$

where $$G_1=\prod_{i=1}^{k} [k(\theta_i)/k(\theta_o)]^{n_i},$$

$$l_{1,0}=\prod_{i=1}^{k} w_i^{n_i} \text{ and } w_i=y_i/\phi_i$$

Hence, noting section 2, we get the distribution of

$$-2 \log (l_3/G_1) \text{ as } x^2$$

with $2k$ degrees of freedom.

Similarly, if we require k to be greater than one, we can show that the non-central distribution of l_4, the likelihood ratio test for testing the hypothesis $H_0 : (\theta_1 = \theta_2 \ldots = \theta_k)$ is the same as that obtained in section 3, for

$$l_4 = \prod_{i=1}^{4} (y_i/y)^{n_i} = l_2,$$

where $y = max\ (y_i, \ldots, y_k)$

References

(1) Hogg, R.V. (1956) : 'On the distribution of the likelihood ratio', Annals of Mathematical Statistics, Vol. 27, pp. 529-532.

(2) Kendall, M.G. and Stuart, Alan : 'The advanced theory of Statistics' Vol. II, p. 236. Charles Griffin and Company Ltd., London.

(3) Wilks, S.S. : 'The large-sample distribution of the likelihood ratio for testing composite hypothesis'. Ann. Math. Stat. Vol. 9. (1938), pp. 60-62.